Coloring k-trees with forbidden monochrome or rainbow triangles

Julian Allagan & Vitaly Voloshin

Department of Mathematics,
University of North Georgia, Watkinsville, Georgia
email: julian.allagan@ung.edu

&

Department of Mathematics and Geomatics,
Troy University, Troy, Alabama
email: vvoloshin@troy.edu

Abstract

An $(\mathscr{H}, H)$-*good* coloring is the coloring of the edges of a (hyper)graph $\mathscr{H}$ such that no subgraph $H \subseteq \mathscr{H}$ is monochrome or rainbow. Similarly, we define an $(\mathscr{H}, H)$-*proper* coloring being the coloring of the vertices of $\mathscr{H}$ with forbidden monochromatic and rainbow copies of H. An $(\mathscr{H}, K_t)$-good coloring is also known as a mixed Ramsey coloring when $\mathscr{H} = K_n$ is a complete graph, and an $(\mathscr{H}, \overline{K}_t)$-proper coloring is a mixed hypergraph coloring of a t-uniform hypergraph $\mathscr{H}$. We highlight these two related theories by finding the number of (T_k^n, K_3)-good and proper colorings for some k-trees, T_k^n with $k \geq 2$. Further, a partition of an edge/vertex set into i nonempty classes is called *feasible* if it is induced by a good/proper coloring using i colors. If r_i is the number of feasible partitions for $1 \leq i \leq n$, then the vector $(r_1, \ldots, r_n)$ is called the *chromatic spectrum*. We investigate and compare the exact values in the chromatic spectrum for some 2-trees, given (T_2^n, K_3)-good versus (T_2^n, K_3)-proper colorings. In particular, we found that when G is a fan, r_2 follows a Fibonacci recurrence.

Keywords: chromatic spectrum, Stirling numbers, mixed hypergraphs, k-trees.

1 Preliminaries

It is customary to define a *hypergraph* $\mathscr{H}$ to be the ordered pair $(X, \mathcal{E})$, where X is a finite set of vertices with *order* $|X| = n$ and $\mathcal{E}$ is a collection of nonempty subsets of X, called *(hyper)edges*. $\mathscr{H}$ is said to be *linear* (otherwise it is *nonlinear*) if $E_1 \cap E_2$ is either empty or a singleton, for any pair of hyperedges. The number of vertices contained in E of $\mathcal{E}$, denoted $|E|$, is the *size* of E. When $|E| = r$, $\mathscr{H}$ is said to be r-*uniform* and a 2-uniform hypergraph

$\mathscr{H} = G$ is a *graph*. For more basic definitions of graphs and hypergraphs, we recommend [17].

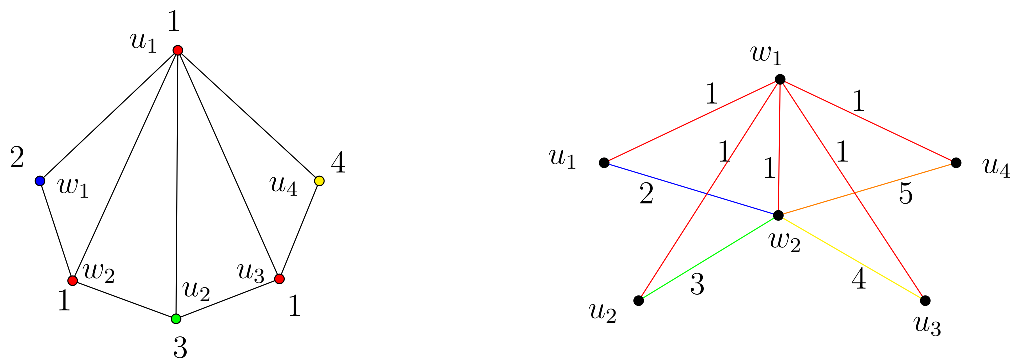
Consider the mapping $c : A \to \{1, 2, \ldots, \lambda\}$ being a λ-coloring of the elements of A. A subset $B \subseteq A$ is said to be *monochrome* if all of its elements share the same color and B is *rainbow* if all of its elements have distinct colors. Let H be a subgraph of a graph G. An edge coloring of G is called $(G; H)$-*good* if it admits no monochromatic copy of H and no rainbow copy of H. Likewise, a $(G; H)$-*proper* coloring is the coloring of the vertices of G such that no copy of H is monochrome or rainbow. Figure 1(A) is an example of a $(G; K_3)$-proper coloring while Figure 1(B) shows a $(G; K_3)$-good coloring.

Axenovich et al.[2] have referred to (K_n, K_3)-good coloring as mixed-Ramsey coloring, a hybrid of classical Ramsey and anti-Ramsey colorings [2, 8, 14] and the minumum and maximum numbers of colors used in a (K_n, K_3)-good coloring have been the subject of extensive research in [2, 3], for instance. Further, in mixed hypergraph colorings [16], a hypergraph $\mathscr{H}$ that admits an $(\mathscr{H}; H)$-proper coloring is called a *bihypergraph* when $H = \overline{K_t}$, the complement of a complete graph on $t \geq 3$ vertices. We note here that, mixed hypergraphs are often used to encode partitioning constraints, and recently bihypergraphs have appeared in communication models for cyber security [11]. Although this paper focuses on graphs, it is worth noting that the results concern some linear and nonlinear bihypergraphs as well.

A partition of an edge/vertex set into i nonempty classes is called *feasible* if it is induced by a good/proper coloring using i colors. If r_i is the number of feasible partitions for each $1 \leq i \leq n$, then the vector $(r_1, \ldots, r_n)$ is called the *chromatic spectrum*. The chromatic spectrum of mixed hypergraphs has been well studied by several researchers such as Kràl and Tuza [5, 6, 12, 13]. Here, we found the values in the chromatic spectrum for any $(G; H)$-good or $(G; H)$-proper colorings when G is some non-isomorphic 2-trees, which are triangulated graphs, and H is a triangle. A comparative analysis of these values is presented in our effort to establish some bounds. In the process, we found that when G is a fan, r_2 follows a shifted Fibonacci recurrence. If we denote the falling factorial by $\lambda^{\underline{i}} = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-i+1)$, then the *(chromatic) polynomial* $P(G; H, \lambda) = P(G; H) = \displaystyle\sum_{i=1}^{n} r_i \lambda^{\underline{i}}$, counts the number of colorings given some constraint on H, using at most λ colors. This polynomial is commonly known in the case of vertex colorings of graphs with a forbidden monochrome subgraph $H \in \{K_2, \overline{K_t}\}$ [4, 7, 15]. In this paper, we also presented this polynomial for k-trees with forbidden monochrome or rainbow K_t for all $t \geq 3$. Here, the Stirling number of the second kind is denoted by $\left\{ {n \atop k} \right\}$; it counts the number of partitions of a set of n elements into k nonempty subsets. See Table 4 for some of its values. These notations and other combinatorial identities can be found in [10]. In Appendix, we present some arrays of the values of the parameters involved in this article; the zero entries are omitted in each table.

2 Chromatic polynomial of some k-trees

As a generalization of a tree, a k-*tree* is a graph which arises from a k-clique by 0 or more iterations of adding n new vertices, each joined to a k-clique in the old graph; This process generates several non-isomorphic k-trees. Figure 1 shows two non-isomorphic 2-trees on 6 vertices. K-trees, when $k \geq 2$, are shown to be useful in constructing reliable network in [9]. Here, we denote by T_k^n, a k-tree on $n+k$ vertices which is obtained from a k-clique S, by repeatedly adding n new vertices and making them adjacent to all the vertices of S. When $k=2$, this particular 2-tree is also known as an $(n+1)$-bridge $\theta(1,2,\ldots,2)$. See Figure 1(B) when $n=4$.



(A) Fan graph, F^4 (B) 5-bridge graph, T_2^4

Figure 1: Two non-isomorphic 2-trees with a unique $(F^4; K_3)$-proper 4-coloring and a unique $(T_2^4; K_3)$-good 5-coloring

Theorem 2.1. *The number of $(T_k^n; K_{k+1})$-good colorings of a k-tree on $n+2$ vertices is* $P(T_k^n; K_{k+1}) = \lambda(\lambda^k - 1)^n + \lambda^{\underline{\binom{k}{2}}}(\lambda^k - (\lambda - \binom{k}{2})^{\underline{k}})^n + (\lambda^{\binom{k}{2}} - \lambda^{\underline{\binom{k}{2}}} - \lambda)\lambda^{nk}$

Proof. Given any coloring of T_k^n, one of the following is true:

(*i*) S is monochromatic giving λ colorings. For each such coloring, there are $\lambda^k - 1$ ways to color the remaining k edges, that arise from each of the n vertices added, giving the first term.

(*ii*) S is rainbow giving $\lambda^{\underline{|S|}}$ colorings. For each such coloring, there are $\lambda^k - (\lambda - |S|)^{\underline{k}}$ ways to color the remaining k edges of each of the $n(k+1)$ cliques, giving the second term.

(*iii*) S is neither monochromatic nor rainbow giving $\lambda^{|S|} - \lambda^{\underline{|S|}} - \lambda$ colorings. For each such coloring, there are λ^k ways to color the remaining edges of each added vertex, giving the last term. The result follows from the fact that $|S| = \binom{k}{2}$. □

Using a similar argument as in the proof of Theorem 2.1 when $|S| = k$, gives

Theorem 2.2. *The number of $(T_k^n; K_{k+1})$-proper colorings of a k-tree on $n+2$ vertices is given by*

$$P(T_k^n; K_{k+1}) = \lambda(\lambda - 1)^n + \lambda^{\underline{k}} k^n + (\lambda^k - \lambda^{\underline{k}} - \lambda)\lambda^n$$

Remark 1: When $k = 2$, observe from the proof of Theorem 2.1 that the number of $(T_2^n; K_3)$-good colorings are identical for non-isomorphic 2-trees. However, in the next section, we show that this is not the case for $(T_2^n; K_3)$-proper colorings.

3 Chromatic spectra of (monochrome and rainbow)-triangle free 2-trees

Here, we find and compare the values in the chromatic spectrum of some 2-trees. The next proposition is instrumental in expressing several formulas in the previous section into a falling factorial form, giving the chromatic spectral values.

Proposition 3.1. *The equality* $\lambda(\lambda-1)^n = \sum_{k=2}^{n+1}\Big[\sum_{s=0}^{n-k+1}(-1)^s\binom{n}{s}\left\{{n+1-s \atop k}\right\}\Big]\lambda^{\underline{k}}$ *holds for all* $n \geq 1$.

Proof. Clearly,

$$\begin{aligned}
\lambda(\lambda-1)^n &= \sum_{i=0}^{n}(-1)^i\binom{n}{i}\lambda^{n+1-i} \\
&= \sum_{i=0}^{n}(-1)^i\binom{n}{i}\Big[\sum_{k=1}^{n+1-i}\left\{{n+1-i \atop k}\right\}\lambda^{\underline{k}}\Big] \\
&= \sum_{k=1}^{n+1}\Big[\sum_{s=0}^{n-k+1}(-1)^s\binom{n}{s}\left\{{n+1-s \atop k}\right\}\Big]\lambda^{\underline{k}} \\
&= \sum_{s=0}^{n}(-1)^s\binom{n}{s}\left\{{n+1-s \atop 1}\right\}\lambda^{\underline{1}} \qquad (1)\\
&+ \sum_{k=2}^{n+1}\Big[\sum_{s=0}^{n-k+1}(-1)^s\binom{n}{s}\left\{{n+1-s \atop k}\right\}\Big]\lambda^{\underline{k}} \qquad (2)
\end{aligned}$$

The result follows from the fact that (1) is equal to zero. □

Corollary 3.0.1. *The chromatic spectrum of any* $(T_2^n; K_3)$*-good coloring is* $(r_2, \ldots, r_k, \ldots, r_{n+1})$, *where* $r_k = 3^n\Big(\sum_{i=0}^{n-k+1}(-1)^i\binom{n}{i}\left\{{n+1-i \atop k}\right\}\Big)$, $k = 2, \ldots, n+1$.

Proof. The result follows from Theorem 2.1 when $k = 2$, and Proposition 3.1. □

Here is the analogous result in a $(G; K_3)$-proper coloring when G is the $(n+1)$-bridge graph $\theta(1, 2, \ldots, 2)$ which was shown to be a 2-tree on $n+2$ vertices. For simplicity, let $T_2'^n = \theta(1, 2, \ldots, 2)$.

Corollary 3.0.2. *The chromatic spectrum of any $(T_2'^n; K_3)$-proper coloring of a 2-tree on $n+2$ vertices is $(r'_2, \ldots, r'_k, \ldots, r'_{n+1})$, where*

$$r'_k = \begin{cases} \sum_{i=0}^{n-k+1} (-1)^i \binom{n}{i} \left\{ {n+1-i \atop k} \right\} & \text{if } k \geq 3 \\ 2^n + 1 & \text{otherwise.} \end{cases}$$

Proof. From Theorem 2.2 (when $k = 2$), we have $P(T_2'^n; K_3) = \lambda(\lambda-1)^n + 2^n \lambda^{\underline{2}}$, to which we apply Proposition 3.1. Also, observe that from (2) when $k = 2$, $\sum_{i=0}^{n-1} (-1)^i \binom{n}{i} \left\{ {n+1-i \atop 2} \right\} = 1$, given the second statement. □

Now, we take a closer look at another well-known 2-tree. Construct a graph G as follows: start with a triangle $\{w_1, w_2, u_1\}$, and iteratively add $n-1$ new vertices, such that each additional vertex u_i is adjacent to the pair $\{u_1, u_{i-1}\}$, for $i = 3, \ldots, n+1$, and u_2 is adjacent to the pair $\{u_1, w_2\}$. We denote G by F^n, a fan on $n+2$ vertices and Figure 1(A) is an example when $n = 4$. Further, from the construction, it is clear that F^n is also a 2-tree. Here, we color the vertices of F^n, and recursively count the number of $(F^n; K_3)$-proper colorings. To help illustrate this recursion, we present the next example.

Example 3.1. *Chromatic spectrum of an $(F^4; K_3)$-proper coloring*

Consider the fan F^4, obtained by iteratively adding $n = 4$ vertices to a base edge $\{w_1, w_2\}$ as shown in Figure 1(A). When $n = 1$, it is clear that there are exactly $2\lambda^{\underline{2}} + \lambda^{\underline{2}}$ ways to color the vertices of the triangle $\{w_1, w_2, u_1\}$ so that it is neither monochrome nor rainbow. The first and second terms count the cases when (a) $c(u_1) \neq c(w_2)$ and (b) $c(u_1) = c(w_2)$, respectively. When $n = 2$, from (a) it follows that for each such colorings, there are exactly two ways to color u_2; either $c(u_2) = c(u_1) \neq c(w_2)$ or $c(u_2) = c(w_2) \neq c(u_1)$. Likewise from (b), there are $\lambda - 1$ ways to color u_2 such that $c(u_2) \neq c(u_1) = c(w_2)$. Together, we have

$$P(F^2; K_3) = 2(2\lambda^{\underline{2}}) + (\lambda-1)\lambda^{\underline{2}} = \lambda^{\underline{2}}[2 + (\lambda-1)] + 2\lambda^{\underline{2}}. \quad (3)$$

As the terms in last expression of (3) are arranged so that the first term counts the case when $c(u_1) \neq c(u_2)$ and the last term counts the case when $c(u_1) = c(u_2)$, we can apply once again the same argument to the newly added vertex u_3. Thus, we have

$$P(F^3; K_3) = 2[\lambda^{\underline{2}}(2 + (\lambda-1))] + (\lambda-1)[2\lambda^{\underline{2}}] = \lambda^{\underline{2}}[2 + 3(\lambda-1)] + \lambda^{\underline{2}}[2 + (\lambda-1)]. \quad (4)$$

Similarly, by adding u_4 to F^3, we obtain from (4),

$$P(F^4; K_3) = \lambda^{\underline{2}}[2 + 5(\lambda-1) + (\lambda-1)^2] + \lambda^{\underline{2}}[2 + 3(\lambda-1)], \quad (5)$$

after rearranging the expression so that the first and last terms count the cases when $c(u_1) \neq c(u_4)$ and $c(u_1) = c(u_4)$, respectively. Hence,

$$P(F^4; K_3) = 4\lambda(\lambda-1) + 8\lambda(\lambda-1)^2 + 1\lambda(\lambda-1)^3. \tag{6}$$

Now apply Proposition 3.1 to each term of (6) to obtain

$$\begin{aligned} P(F^4; K_3) &= 4[\binom{1}{0}\left\{{2 \atop 2}\right\}]\lambda^{\underline{2}} + 8[\binom{2}{0}\left\{{3 \atop 2}\right\} - \binom{2}{1}\left\{{2 \atop 2}\right\}]\lambda^{\underline{2}} \\ &+ 1[\binom{3}{0}\left\{{4 \atop 2}\right\} - \binom{3}{1}\left\{{3 \atop 2}\right\} + \binom{3}{2}\left\{{2 \atop 2}\right\}]\lambda^{\underline{2}} \\ &+ 8[\binom{2}{0}\left\{{3 \atop 3}\right\}]\lambda^{\underline{3}} + 1[\binom{3}{0}\left\{{4 \atop 3}\right\} - \binom{3}{1}\left\{{3 \atop 3}\right\}]\lambda^{\underline{3}} + 1[\binom{4}{0}\left\{{4 \atop 4}\right\}]\lambda^{\underline{4}} \\ &= [4 + 8(3-2) + 1(7 - 3\cdot 3 + 3)]\lambda^{\underline{2}} + [8 + 1(6-3)]\lambda^{\underline{3}} + 1[\lambda^{\underline{4}}] \\ &= 13\lambda^{\underline{2}} + 11\lambda^{\underline{3}} + 1\lambda^{\underline{4}}. \end{aligned} \tag{7}$$

Thus, the chromatic spectrum of any $(F^4; K_3)$-proper coloring is $(13, 11, 1)$.

To support a general recursion presented in the next theorem, we let $a_{4,0} = 2, a_{4,1} = 5, a_{4,2} = 1, a_{4,3} = 2$, and $a_{4,4} = 3$; Table 2 shows the values of each $a_{i,j}$ (when $n = 4$). With these coefficients we obtain directly from (5)

$$\begin{aligned} P(F^4; K_3) &= \lambda^{\underline{2}}[a_{4,0}(\lambda-1)^0 + a_{4,1}(\lambda-1)^1 + a_{4,2}(\lambda-1)^2] \\ &+ \lambda^{\underline{2}}[a_{4,3}(\lambda-1)^0 + a_{4,4}(\lambda-1)^1] \\ &= \phi(4,0)\lambda(\lambda-1)^1 + \phi(4,1)\lambda(\lambda-1)^2 + \phi(4,2)\lambda(\lambda-1)^3, \end{aligned} \tag{8}$$

where

$$\phi(4, r) = \begin{cases} a_{4,r} + a_{4,3+r} & \text{if } r < 2 \\ a_{4,2} & \text{otherwise} \end{cases}$$

We note that (8) follows from Theorem 3.1, when $n = 4$. Now, Proposition 3.1 gives

$$\begin{aligned} P(F^4; K_3) &= [\phi(4,0)(1) + \phi(4,1)(3-2) + \phi(4,2)(7 - 3\cdot 3 + 3)]\lambda^{\underline{2}} \\ &+ [\phi(4,1)(1) + \phi(4,2)(6-3)]\lambda^{\underline{3}} + [\phi(4,2)]\lambda^{\underline{4}} \\ &= [\phi(4,0) + \phi(4,1) + \phi(4,2)]\lambda^{\underline{2}} + [\phi(4,1) + 3\phi(4,2)]\lambda^{\underline{3}} + \phi(4,2)\lambda^{\underline{4}}. \end{aligned} \tag{9}$$

Again, observe that (9) follows from (13) when $n = 4$. The values of $\phi(n, r)$ when $n = 11$ are recorded in Table 3, with $0 \leq r \leq \lfloor \frac{n}{2} \rfloor$. Thus, since $\phi(4, 0) = 4$, $\phi(4, 1) = 8$, $\phi(4, 2) = 1$, we have

$$P(F^4; K_3) = 13\lambda^{\underline{2}} + 11\lambda^{\underline{3}} + 1\lambda^{\underline{4}}.$$

Table 1 in Appendix shows some of the chromatic spectal values given a $(T_2^n; K_3)$-good coloring, a $(T_2'^n; K_3)$-proper coloring and an $(F^n; K_3)$-proper coloring when $n = 1, \ldots, 6$. These values can be derived from Corollary 3.0.1, Corollary 3.0.2, and Corollary 3.1.1 respectively, for each coloring condition.

Theorem 3.1. *The number of $(F^n; K_3)$-proper colorings is*

$$P(F^n;K_3) = \sum_{0\le r\le \lfloor \frac{n}{2}\rfloor} \phi(n,r)\lambda(\lambda-1)^{r+1}, \text{ where}$$

$$\phi(n,r) = \begin{cases} a_{n,r} + a_{n,\lceil \frac{n+1}{2}\rceil + r} & \text{if } r < \frac{n}{2} \\ a_{n,\frac{n}{2}} & \text{otherwise} \end{cases}$$

and the values of $a_{i,j}$ satisfying, for $0 \le j \le i \le n$,

(i) $a_{i,0} = 2$ *and* $a_{1,1} = 1$

(ii) for all even $i \ge 2$, $a_{i,j} = \begin{cases} a_{i-1,j} + a_{i-1,j+\lfloor \frac{i-1}{2}\rfloor} & ; 1 \le j \le \lceil \frac{i-1}{2}\rceil \\ 1 & ; j = \frac{i}{2} \\ a_{i-1,j-\lceil \frac{i+1}{2}\rceil} & ; \lceil \frac{i+1}{2}\rceil \le j \le i \end{cases}$

(iii) for all odd $i \ge 3$, $a_{i,j} = \begin{cases} a_{i-1,j} + a_{i-1,j+\lfloor \frac{i-1}{2}\rfloor} & ; 1 \le j \le \frac{i-1}{2} \\ a_{i-1,j-\lceil \frac{i}{2}\rceil} & ; \lceil \frac{i}{2}\rceil \le j \le i \end{cases}$

Proof. When $n = 1$, it follows that $P(F^1;K_3) = \phi(1,0)\lambda(\lambda-1)^1 = [a_{1,0} + a_{1,1}]\lambda(\lambda-1)^1 = 3\lambda(\lambda-1)$, since $a_{1,0} = 2$ and $a_{1,1} = 1$ by condition (i). For $n \ge 2$, at each iteration, we separate the cases when $c(u_1) \ne c(u_k)$ from when $c(u_1) = c(u_k)$. Further, we rearrange the terms of the resulting expression of $P(F^k;K_3)$ so that the first counts the colorings $c(u_1) \ne c(u_k)$, and the last counts the colorings $c(u_1) = c(u_k)$ for $k = 1,\ldots,n$. Hence, for $n \ge 1$,

$$\begin{aligned} P(F^n;K_3) &= \lambda^2\Big(\sum_{1\le k\le \lceil \frac{n+1}{2}\rceil} a_{n,k-1}(\lambda-1)^{k-1}\Big) + \lambda^2\Big(\sum_{1+\lceil \frac{n+1}{2}\rceil \le k\le n} a_{n,k-1}(\lambda-1)^{k-\lceil \frac{n+1}{2}\rceil - 1}\Big) \\ &= \sum_{1\le k\le \lceil \frac{n+1}{2}\rceil} [a_{n,k-1} + a_{n,\lceil \frac{n+1}{2}\rceil + k-1}]\lambda(\lambda-1)^{k+1}, \end{aligned} \tag{10}$$

where the coefficients $a_{i,j}$ are obtained recursively from items $(i) - (iii)$. By letting $a_{i,j} = 0$ when $i < j$, it follows that

$$P(F^n;K_3) = \sum_{0\le r\le \lfloor \frac{n}{2}\rfloor} \phi(n,r)\lambda(\lambda-1)^{r+1}, \tag{11}$$

where

$$\phi(n,r) = \begin{cases} a_{n,r} + a_{n,\lceil \frac{n+1}{2}\rceil + r} & \text{if } r < \frac{n}{2} \\ a_{n,\frac{n}{2}} & \text{otherwise} \end{cases}$$

□

Observation 1: The previous result can be reinterpreted as follows: Let $a_{0,0} = 2$ and define an $(n+1)\times(n+1)$ matrix A whose entries are the coefficients $a_{i,j}$ for $0 \le i, j \le n$. It follows that (10) is equivalent to the equation $P = \lambda A \cdot B$, where

$$P = \begin{bmatrix} P(F^0;K_3)+\lambda(\lambda-2) \\ P(F^1;K_3) \\ \vdots \\ P(F^n;K_3) \end{bmatrix},\ A = \begin{bmatrix} a_{0,0} & & & \\ a_{1,0} & a_{1,1} & & \\ \vdots & \vdots & \ddots & \\ a_{n,0} & a_{n,1} & \dots & a_{n,n} \end{bmatrix}$$

$$B = \left[B^1|B^2\right],^T \textit{ with } B^1 = \left[(\lambda-1)^1 \dots (\lambda-1)^{\lceil\frac{n+1}{2}\rceil}\right] \textit{ and } B^2 = \left[(\lambda-1)^1 \dots (\lambda-1)^{\lfloor\frac{n+1}{2}\rfloor}\right].$$

When $n = 10$, we present the entries of the lower triangular matrix A in Table 2 to help in the verification of the formula. The matrix A has several interesting properties some of which we discuss in the next observation. For now, it is easy to see that its determinant

$$det(A) = \prod_{i=0}^{n} a_{i,i} = 2(\lceil\frac{n+1}{2}\rceil)!$$

and its characteristic polynomial is given by

$$(-1)^{n+1}(x-1)^{\lceil\frac{n}{2}\rceil}(x-2)^2(x-3)\dots(x-\lceil\frac{n+1}{2}\rceil)$$

Corollary 3.1.1. *The values in the chromatic spectrum of any $(F^n;K_3)$-proper coloring are given by* $r''_k = \sum_{k-2\le r\le\lfloor\frac{n}{2}\rfloor} \phi(n,r)\Big(\sum_{0\le i\le r-k+2}(-1)^i\binom{r+1}{i}\left\{ {r+2-i \atop k} \right\}\Big)$, *for each* $k = 2,\dots,\lceil\frac{n+1}{2}\rceil+1$, *with*

$$\phi(n,r) = \begin{cases} a_{n,r} + a_{n,\lceil\frac{n+1}{2}\rceil+r} & \textit{if } r < \lfloor\frac{n}{2}\rfloor \\ a_{n,\lfloor\frac{n}{2}\rfloor} & \textit{otherwise} \end{cases}$$

Proof. For each $r = 0,\dots,\lfloor\frac{n}{2}\rfloor$, we apply Proposition 3.1 to $P(F^n;K_3)$, giving

$$\begin{aligned} P(F^n;K_3) &= \phi(n,0)[(-1)^0\binom{1}{0}\left\{ {2 \atop 2} \right\}]\lambda^{\underline{2}} \\ &+ \phi(n,1)[(-1)^0\binom{2}{0}\left\{ {3 \atop 2} \right\} + (-1)^1\binom{2}{1}\left\{ {2 \atop 2} \right\}]\lambda^{\underline{2}} + \phi(n,1)[(-1)^0\binom{2}{0}\left\{ {3 \atop 3} \right\}]\lambda^{\underline{3}} \\ &+ \phi(n,2)[(-1)^0\binom{3}{0}\left\{ {4 \atop 2} \right\} + (-1)^1\binom{3}{1}\left\{ {3 \atop 2} \right\} + (-1)^2\binom{3}{2}\left\{ {2 \atop 2} \right\}]\lambda^{\underline{2}} \\ &+ \phi(n,2)[(-1)^0\binom{3}{0}\left\{ {4 \atop 3} \right\} + (-1)^1\binom{3}{1}\left\{ {4 \atop 4} \right\}]\lambda^{\underline{3}} \end{aligned}$$

$$+ \quad \phi(n,3)[(-1)^0\binom{3}{0}\left\{{4 \atop 4}\right\}]\lambda^{\underline{4}}$$

$$\vdots$$

$$+ \quad \phi(n,\lfloor\frac{n}{2}\rfloor)[(-1)^0\binom{\lceil\frac{n+1}{2}\rceil}{0}\left\{{\lceil\frac{n+1}{2}\rceil+1 \atop \lceil\frac{n+1}{2}\rceil+1}\right\}]\lambda^{\underline{\lceil\frac{n+1}{2}\rceil+1}}. \tag{12}$$

Therefore,

$$P(F^n;K_3) \;=\; \sum_{k=2}^{\lceil\frac{n+1}{2}\rceil+1}\Big(\sum_{r=k-2}^{\lfloor\frac{n}{2}\rfloor}\phi(n,r)\Big[\sum_{0\le i\le r-k+2}(-1)^i\binom{r+1}{i}\left\{{r+2-i \atop k}\right\}\Big]\Big)\lambda^{\underline{k}}, \tag{13}$$

giving the result.

□

Observation 2: When $k=\lceil\frac{n+1}{2}\rceil+1$, the last term of (13) is

$$\phi(n,\lfloor\tfrac{n}{2}\rfloor)=\begin{cases}1 & \text{if n is even}\\ 3+\frac{n-1}{2} & \text{otherwise}\end{cases}$$

Also, it is worth noting that when $k=2$, $\sum_{r=0}^{\lfloor\frac{n}{2}\rfloor}\phi(n,r)\Big[\sum_{0\le i\le r}(-1)^i\binom{r+1}{i}\left\{{r+2-i \atop 2}\right\}\Big] = \sum_{r=0}^{\lfloor\frac{n}{2}\rfloor}\phi(n,r)$; this proceeds from the simple fact that $\sum_{i=0}^{n}(-1)^i\binom{n}{i}\left\{{n+1-i \atop 2}\right\}=1$, for all n. Further, observe that if we define $b_i=\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\phi(n,j)$ for each $i\le n$, it follows that $b_i=\sum_j a_{i,j}$ and the sequence $\{b_n\}$ satisfies the shifted Fibonacci recurrence given by $b_0=2$, $b_1=3$ and $b_n=b_{n-1}+b_{n-2}$, for $n\ge 2$. From this observation, we determine the generating function in the next proposition.

Proposition 3.2. *The number of partitions of the $n+2$ vertices of a fan into 2 nonempty classes such that no triangle is monochrome or rainbow is given by*

$$b_n=\frac{1}{\sqrt{5}}[(2+\sqrt{5})\alpha^n-(2-\sqrt{5})\beta^n],\ \textit{where}\ \alpha=\frac{1+\sqrt{5}}{2}\ \textit{and}\ \beta=\frac{1-\sqrt{5}}{2}$$

Proof. Let $b(x) = \sum_{n=0}^{\infty} b_n x^n$ such that $b_0 = 2$, $b_1 = 3$ and $b_n = b_{n-1} + b_{n-2}$. It follows that

$$\begin{aligned} b(x) &= 2 + 3x + \sum_{n=2}^{\infty} b_n x^n \\ &= 2 + 3x + x\sum_{k=1}^{\infty} b_k x^k + x^2 \sum_{k=0}^{\infty} b_k x^k \\ &= 2 + 3x + x(\sum_{k=0}^{\infty} b_k x^k - 2) + x^2 \sum_{k=0}^{\infty} b_k x^k \\ &= 2 + x + xb(x) + x^2 b(x). \end{aligned}$$

This implies that $b(x) = \frac{2+x}{1-x-x^2} = -\frac{2+x}{(x+\alpha)(x+\beta)}$, with $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Using a partial fraction decomposition, and subsequently the power series, we obtain

$$\begin{aligned} b(x) &= \frac{1}{\sqrt{5}}\Big[\frac{\beta-2}{x+\beta} - \frac{\alpha-2}{x+\alpha}\Big] \\ &= \frac{1}{\sqrt{5}}\Big[\frac{\beta-2}{\beta}(\sum_{n=0}^{\infty} \alpha^n x^n) - \frac{\alpha-2}{\alpha}(\sum_{n=0}^{\infty} \beta^n x^n)\Big] \\ &= \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}}\Big[\frac{\beta-2}{\beta}\alpha^n - \frac{\alpha-2}{\alpha}\beta^n\Big] x^n, \end{aligned}$$

giving that $b_n = \frac{1}{\sqrt{5}}\Big[\frac{\beta-2}{\beta}\alpha^n - \frac{\alpha-2}{\alpha}\beta^n\Big]$. The result follows, after a simplification.

□

In summary, the extreme chromatic spectral values given the aforementioned colorings are clear; the lower values are, $r_2 = 3^n$, $r'_2 = 2^n + 1$, $r''_2 = b(x)$ where $b(x) = \frac{1}{\sqrt{5}}\Big[\frac{\beta-2}{\beta}\alpha^n - \frac{\alpha-2}{\alpha}\beta^n\Big]$. Also, for all $n > 1$, the upper values are also shown to be $r_{n+1} = 3^n$, $r'_{n+1} = 1$, and $r''_{\lceil \frac{n+1}{2} \rceil + 1} = \begin{cases} 1 & \text{if n even} \\ 3 + \frac{n-1}{2} & \text{otherwise} \end{cases}$

4 Conclusion and future work

To the best of our knowledge, the problem of finding the exact chromatic spectral values in a (K_n, K_t)-good coloring remains open for all $t \geq 3$ and larger values of n; this particular problem which was originally by one of the authors has greatly inspired this research. When G is a 2-tree, the findings in Corollaries 3.0.1, 3.0.2, and 3.1.1 suggest the existence of some constant $c < 1$, such that $r_k^* = cr_k$ where r_k^* and r_k are the corresponding values in the chromatic spectra of a $(G; K_3)$-proper and a $(G; K_3)$-good coloring, respectively. For instance, $c = (\frac{1}{3})^n$ when G is an $(n+1)$-bridge. Further work is needed to determine whether the values in the chromatic spectrum of a $(G; H)$-good coloring remain upper bounds for their counterparts in a $(G; H)$-proper coloring, given any other graph G and some subgraph H.

Also, the original definition of a $(G; H)$-proper coloring can be extended to include more than one subgraph. For instance, a $(G; H_1, \ldots, H_m)$-proper coloring is the coloring of the vertices of G such that no copy of (distinct) subgraphs H_i is monochrome or rainbow, for $i = 1, \ldots, m$. As such, when $G = \mathscr{H}$ and $H_i = \overline{K}_{t_i}$, $\mathscr{H}$ is a non-uniform bihypergraph with hyperedges of size $t_i \geq 3$. Some related results concerning non-uniform bihypergraphs can be found in [1]. As a step in this direction for graphs, we propose the next lemma. This lemma shows that the chromatic spectral values of any $(F^n; K_3, H)$-proper coloring are identical when $H \in \{K_{1,t}, C_4, P_3 \Box P_2\}$, where $P_3 \Box P_2$ is isomorphic to $\theta(1, 3, 3)$, and $K_{1,t}$ is a complete bipartite graph with parts sizes 1 and $t \geq 2$.

Lemma 4.1. *Any (monochrome and rainbow)-triangle free proper coloring of a fan on $n+2$ vertices is an $(F^n; K_3, H)$-proper coloring for each $H \in \{K_{1,t}, C_4, P_3 \Box P_2\}$, with $\lfloor \frac{n+1}{2} \rfloor \leq t \leq n+1$.*

Proof. Let $S = \{w_1, w_2, u_2, u_3, \ldots, u_n\}$ denote the set of rim vertices and let $S_1 = \{w_1, u_2, \ldots, u_{2r}\}$, for each $1 \leq r \leq \lfloor \frac{n}{2} \rfloor$. Suppose $H = K_{1,t}$ and consider a coloring of F^n such that $c(u_1) = c(v_1)$ for each $v_1 \in S_1$. If F^n contains no monochrome and rainbow triangle, it must be that $c(u_1) \neq c(v_2)$ for each vertex $v_2 \in (S \setminus S_1)$. Pick any $v_2' \notin S_1$, and by letting $S_1 \cup \{u_1, v_2'\}$ be the vertex set of the subgraph $K_{1,t} \subset F^n$, it is clear that $K_{1,t}$ is neither monochrome nor rainbow and the size of $S_1 \cup \{v_2'\}$ gives the lower bound of t. To obtain the upper bound of t, color each vertex $v \in S$ with the same color and let $c(u_1) \neq c(v)$ for each $v \in S$. This gives an $(F^n; K_3)$-proper coloring and it is also an $(F^n; K_3, K_{1,t})$-proper coloring, where the vertex set of $K_{1,t} \subset F^n$ is $S \cup \{u_1\}$.

Now we show that any $(F^n; K_3)$-proper coloring is an $(F^n; K_3, C_4)$-proper coloring. Since every cycle on 4 vertices $C_4 \subset F^n$ must include u_1, assume that $C_4 = (u_1, v_1, v_2, v_3, u_1)$, an ordered sequence of vertices. If the set $\{u_1, v_1, v_2, v_3\}$ is monochrome/rainbow, then $C_4 \subset F^n$ contains a monochrome/rainbow triangle, which is impossible. Hence C_4 is neither monochrome nor rainbow, giving an $(F^n; K_3, C_4)$-proper coloring.

For all $n \geq 5$, observe that $H = P_3 \Box P_2 \subset F^n$, and the argument follows from the fact that $C_4 \subset P_3 \Box P_2$. □

In conclusion, it is worth noting that future work can address the coloring of the vertices of a graph with either forbidden monochrome subgraphs or forbidden rainbow subgraphs (but not both). As a step in this direction, we present a simple case when coloring the elements of an n-set such that no t-subset is rainbow.

Corollary 4.0.2. *The chromatic spectral values in the colorings of the vertices of a complete graph K_n such that no K_t is rainbow are given by $r_k = \left\{ {n \atop k} \right\}$, for $k = 1, \ldots, t-1$.*

Note that these values also correspond to the chromatic spectral values of any complete t-uniform cohypergraph of order n; cohypergraphs are hypergraphs whose hyperedges are forbidden to be rainbow given any proper (vertex) coloring [16].

References

[1] J. Allagan, D. Slutzky, *Chromatic Polynomials of Mixed Hypergraphs*, Australasian Journal of Combinatorics **58(1)** (2014), 197-213.

[2] M. Axenovich, J. Choi, *A note on monotonicity of mixed Ramsey numbers*, Discrete Math. **311** (2011), 2020-2023.

[3] M. Axenovich, P. Iverson, *Edge-colorings avoiding rainbow and monochromatic subgraphs*, Discrete Math. 3**08** (2008), no. 20, 4710-4723.

[4] M. Borowiecki, E. łazuka, *Chromatic polynomials of hypergraphs*, Discuss Math., Graph Theory **20(2)** (2000), 293-301.

[5] C. Bujtàs, Zs. Tuza, *Uniform mixed hypergraphs: the possible numbers of colors*, Graphs Combin. **24** (2008), 1-12.

[6] K. Diao, K. Wang, P. Zhao, *The chromatic spectrum of 3-uniform bi-hypergraphs*, Discrete Math. **3(11)** (2011), 650-656.

[7] F.M. Dong, K.M. Koh and K.L. Teo, Chromatic polynomials and chromaticity of graphs, Singapore: World Scientific Publishing. xxvii, 2005.

[8] P. Erdös, M. Simonovits and V. T. Sòs, *Anti-Ramsey theorems*, Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdös on his 60th birthday), Vol. II, pp. 633-643. Colloq. Math. Soc. Janos Bolyai, Vol. 10, North-Holland, Amsterdam, 1975.

[9] A.M. Farley, *Networks Immune to Isolated Failures*, Networks **11** (1981), pp. 225-268.

[10] R.L. Graham, D.E. Knuth and O. Patashnik, *Concrete Mathematics*, Addison–Wesley Publishing, 1994, pp.264–265.

[11] A. Jaffe, T. Moscibroda and S. Sen, *On the price of equivocation in Byzantine agreement*, Proc. 31^{st} Principles of Distributed Computing (PODC), 2012.

[12] T. Jiang, D. Mubayi, Zs. Tuza, V. Voloshin and D. West, *The Chromatic Spectrum of Mixed Hypergraphs*, Graphs and Combin. **18** (2002), 309-318.

[13] D. Kràl', *On Feasible Sets of Mixed Hypergraphs*, Electron. J. of Combin. **11 (1)** (2004), R19.

[14] F. P. Ramsey, *On a Problem of Formal Logic*, Proc. of the London Math. Soc. **30** (1930), 264-286.

[15] I. Tomescu, *Chromatic coefficients of linear uniform hypergraphs*, J. Combin. Theory B **72** (1998), 229-235.

[16] V. I. Voloshin, *Coloring Mixed Hypergraphs: Theory, Algorithms and Applications*, American Mathematical Society, 2002.

[17] D. West, *Introduction to Graph Theory*, Prentice Hall, 2001.

Appendix

	$(T_2^n;K_3)$-good	$(T_2'^n;K_3)$-proper	$(F^n;K_3)$-proper
$n=1$	**(3)**	**(3)**	**(3)**
$n=2$	3^2**(1,1)**	**(5,1)**	**(5,1)**
$n=3$	3^3**(1,3,1)**	**(9,3,1)**	**(8,4)**
$n=4$	3^4**(1,7,6,1)**	**(17,7,6,1)**	**(13,11,1)**
$n=5$	3^5**(1,15,25,10,1)**	**(33,15,25,10,1)**	**(27,17,5)**
$n=6$	3^6**(1,31,90,65,15,1)**	**(65,31,90,65,15,1)**	**(37,62,7,1)**

Table 1: chromatic spectral values of some $(G;K_3)$-good colorings and some $(G;K_3)$-proper colorings for $n \leq 6$

$n \backslash j$	0	1	2	3	4	5	6	7	8	9	10	11
0	**2**											
1	**2**	**1**										
2	**2**	**1**	**2**									
3	**2**	**3**	**2**	**1**								
4	**2**	**5**	**1**	**2**	**3**							
5	**2**	**7**	**4**	**2**	**5**	**1**						
6	**2**	**9**	**9**	**1**	**2**	**7**	**4**					
7	**2**	**11**	**16**	**5**	**2**	**9**	**9**	**1**				
8	**2**	**13**	**25**	**14**	**1**	**2**	**11**	**16**	**5**			
9	**2**	**15**	**36**	**30**	**6**	**2**	**13**	**25**	**14**	**1**		
10	**2**	**17**	**49**	**55**	**20**	**1**	**2**	**15**	**36**	**30**	**6**	
11	**2**	**19**	**64**	**91**	**50**	**7**	**2**	**17**	**49**	**55**	**20**	**1**

Table 2: Table of values of $a_{i,j}$, which are the entries of the matrix A when $n=11$

$n\backslash r$	0	1	2	3	4	5
0	**2**					
1	**3**					
2	**4**	**1**				
3	**4**	**4**				
4	**4**	**8**	**1**			
5	**4**	**12**	**5**			
6	**4**	**16**	**13**	**1**		
7	**4**	**20**	**25**	**6**		
8	**4**	**24**	**41**	**19**	**1**	
9	**4**	**28**	**61**	**44**	**7**	
10	**4**	**32**	**85**	**85**	**26**	**1**
11	**4**	**36**	**113**	**146**	**70**	**8**

Table 3: Table of values of $\phi(n, r)$ when $n = 11$

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
0	**1**											
1	**0**	**1**										
2	**0**	**1**	**1**									
3	**0**	**1**	**3**	**1**								
4	**0**	**1**	**7**	**6**	**1**							
5	**0**	**1**	**15**	**25**	**10**	**1**						
6	**0**	**1**	**31**	**90**	**65**	**15**	**1**					
7	**0**	**1**	**63**	**301**	**350**	**140**	**21**	**1**				
8	**0**	**1**	**127**	**966**	**1701**	**1050**	**266**	**28**	**1**			
9	**0**	**1**	**255**	**3025**	**7770**	**6951**	**2646**	**462**	**36**	**1**		
10	**0**	**1**	**511**	**9330**	**34105**	**42525**	**22827**	**5880**	**750**	**45**	**1**	
11	**0**	**1**	**1023**	**2850**	**145750**	**246730**	**179487**	**63987**	**11880**	**1155**	**55**	**1**

Table 4: Table of values of $\left\{ {n \atop k} \right\}$ when $n = 11$